varies from a to $a + \omega_1$. It is consequently an integer. The same applies to the other pair of opposite sides. Therefore the value of (8) is of the form $n_1\omega_1 + n_2\omega_2$, and the theorem is proved.

3. THE WEIERSTRASS THEORY

The simplest elliptic functions are of order 2, and such functions have either a double pole with residue zero, or two simple poles with opposite residues. We shall follow the classical example of Weierstrass, who chose a function with a double pole as the starting point of a systematic theory.

3.1. The Weierstrass $\wp$-function. We may as well place the pole at the origin, and since multiplication with a constant factor is clearly irrelevant, we may require that the singular part is z^{-2}. If f is elliptic and has only this singularity at the origin and its congruent points, it is easy to see that f must be an even function. Indeed, $f(z) - f(-z)$ has the same periods and no singularity. Therefore it must reduce to a constant, and on setting $z = \omega_1/2$ we conclude that the constant is zero.

A constant can be added at will, and we can therefore choose the constant term in the Laurent development about the origin to be zero. With this additional normalization $f(z)$ is uniquely determined, and it is traditionally denoted by a special typographical symbol $\wp(z)$. The Laurent development has the form

$$\wp(z) = z^{-2} + a_1z^2 + a_2z^4 + \cdots .$$

So far all this is hypothetical, for we have not yet shown the existence of an elliptic function with this development. We shall follow the usual procedure in such cases, namely to postulate the existence and derive an explicit expression. The clue is to develop in partial fractions by the method in Chap. 5, Sec. 2. Our aim is to prove the formula

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \tag{9}$$

where the sum ranges over all $\omega = n_1\omega_1 + n_2\omega_2$ except 0. Observe that $(z - \omega)^{-2}$ is the singular part at ω, and that we have subtracted ω^{-2} in order to produce convergence.

Our first task is to verify that the series converges. If $|\omega| > 2|z|$, say, an immediate estimate gives

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{z(2\omega - z)}{\omega^2(z-\omega)^2} \right| \leqq \frac{10|z|}{|\omega|^3}.$$

Therefore the series (9) converges, uniformly on every compact set, provided that

$$\sum_{\omega\neq 0} \frac{1}{|\omega|^3} < \infty.$$

This is indeed the case. Because ω_2/ω_1 is nonreal, there exists a $k > 0$ such that $|n_1\omega_1 + n_2\omega_2| \geqq k(|n_1| + |n_2|)$ for all real pairs (n_1,n_2). If we consider only integers there are $4n$ pairs (n_1,n_2) with $|n_1| + |n_2| = n$. This gives

$$\sum_{\omega\neq 0} |\omega|^{-3} \leqq 4k^{-3}\sum_1^\infty n^{-2} < \infty.$$

The next step is to prove that the right-hand side of (9) has periods ω_1 and ω_2. Direct verification is relatively cumbersome. Instead we write, temporarily,

$$f(z) = \frac{1}{z^2} + \sum_{\omega\neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) \tag{10}$$

and obtain by termwise differentiation

$$f'(z) = -\frac{2}{z^3} - \sum_{\omega\neq 0}\frac{2}{(z-\omega)^3} = -2\sum_{\omega}\frac{1}{(z-\omega)^3}.$$

The last sum is obviously doubly periodic. Therefore $f(z + \omega_1) - f(z)$ and $f(z + \omega_2) - f(z)$ are constants. Because $f(z)$ is even (as seen from (10)), it suffices to choose $z = -\omega_1/2$ and $z = -\omega_2/2$ to conclude that the constants are zero. We have thus proved that f has the asserted periods.

It follows now that $\wp(z) - f(z)$ is a constant, and by the form of the development at the origin the constant is zero. We have thereby proved the existence of $\wp(z)$, and also that it can be represented by the series (9). For convenient reference we display the important formula

$$\wp'(z) = -2\sum_{\omega}\frac{1}{(z-\omega)^3}. \tag{11}$$

3.2. The Functions $\zeta(z)$ and $\sigma(z)$. Because $\wp(z)$ has zero residues, it is the derivative of a single-valued function. It is traditional to denote the antiderivative of $\wp(z)$ by $-\zeta(z)$, and to normalize it so that it is odd. By use of (9) we are led to the explicit expression

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\neq 0}\left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right). \tag{12}$$

The convergence is obvious, for apart from the term $1/z$ we obtain the new series by integration from 0 to z along any path that does not pass through the poles.

It is clear that $\zeta(z)$ satisfies conditions $\zeta(z + \omega_1) = \zeta(z) + \eta_1$, $\zeta(z + \omega_2) = \zeta(z) + \eta_2$, where η_1 and η_2 are constants. They are connected with ω_1, ω_2 by a very simple relation. To derive it we choose any $a \neq 0$ and observe that

$$\frac{1}{2\pi i} \int_{\partial P_a} \zeta(z)\, dz = 1,$$

by the residue theorem. The integral is easy to evaluate by adding the contributions from opposite sides of the parallelogram, and we obtain the equation

$$\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i,$$

known as *Legendre's relation.*

The integration can be carried one step further provided that we use an exponential to eliminate the multiple-valuedness. Just as easily we can verify directly that the product

$$\sigma(z) = z \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{z/\omega + \frac{1}{2}(z/\omega)^2} \tag{13}$$

converges and represents an entire function which satisfies

$$\sigma'(z)/\sigma(z) = \zeta(z).$$

The formula (13) is a canonical product representation of $\sigma(z)$.

How does $\sigma(z)$ change when z is replaced by $z + \omega_1$ or $z + \omega_2$? From

$$\frac{\sigma'(z + \omega_1)}{\sigma(z + \omega_1)} = \frac{\sigma'(z)}{\sigma(z)} + \eta_1$$

it follows at once that

$$\sigma(z + \omega_1) = C_1\sigma(z)e^{\eta_1 z}$$

with constant C_1. To determine the constant we observe that $\sigma(z)$ is an odd function. On setting $z = -\omega_1/2$ the value of C_1 can be determined, and we find that $\sigma(z)$ satisfies

$$\begin{aligned} \sigma(z + \omega_1) &= -\sigma(z)e^{\eta_1(z+\omega_1/2)} \\ \sigma(z + \omega_2) &= -\sigma(z)e^{\eta_2(z+\omega_2/2)}. \end{aligned} \tag{14}$$

EXERCISES

1. Show that any even elliptic function with periods ω_1, ω_2 can be expressed in the form

$$C \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)} \qquad (C = \text{const.})$$

provided that 0 is neither a zero nor a pole. What is the corresponding form if the function either vanishes or becomes infinite at the origin?

2. Show that any elliptic function with periods ω_1, ω_2 can be written as

$$C \prod_{k=1}^{n} \frac{\sigma(z - a_k)}{\sigma(z - b_k)} \qquad (C = \text{const.}).$$

Hint: Use (14) and Theorem 6.

3.3. The Differential Equation. By use of formula (12) it is easy to derive the Laurent expansion of $\zeta(z)$ about the origin, and differentiation will then yield the corresponding expansion of $\wp(z)$. We have first

$$\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} = -\frac{z^2}{\omega^3} - \frac{z^3}{\omega^4} - \cdots$$

and when we sum over all periods we obtain

$$\zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty} G_k z^{2k-1}$$

where we have written

$$G_k = \sum_{\omega \neq 0} \frac{1}{\omega^{2k}}.$$

Observe that the corresponding sums of odd powers of the periods are zero, as was to be expected since ζ is an odd function. Because

$$\wp(z) = -\zeta'(z)$$

we obtain further

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} (2k - 1)G_k z^{2k-2}.$$

In the following computation we write down only the significant terms, since it is understood that the omitted terms are of higher order:

$$\wp(z) = \frac{1}{z^2} + 3G_2 z^2 + 5G_3 z^4 + \cdots$$

$$\wp'(z) = -\frac{2}{z^3} + 6G_2 z + 20G_3 z^3 + \cdots$$

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{24G_2}{z^2} - 80G_3 + \cdots$$

$$4\wp(z)^3 = \frac{4}{z^6} + \frac{36G_2}{z^2} + 60G_3 + \cdots$$

$$60G_2\wp(z) = \frac{60G_2}{z^2} + 0 + \cdots.$$

The last three lines yield

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2\wp(z) = -140G_3 + \cdots.$$

Here the left-hand side is a doubly periodic function, and the right-hand side has no poles. We may therefore conclude that

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_2\wp(z) - 140G_3.$$

It is customary to set $g_2 = 60G_2$, $g_3 = 140G_3$ so that the equation becomes

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{15}$$

This is a first-order differential equation for $w = \wp(z)$. It can be solved explicitly, namely, by the formula

$$z = \int^w \frac{dw}{\sqrt{4w^3 - g_2w - g_3}} + \text{constant},$$

which shows that $\wp(z)$ is the inverse of an elliptic integral. More accurately, this connection is expressed by the identity

$$z - z_0 = \int_{\wp(z_0)}^{\wp(z)} \frac{dw}{\sqrt{4w^3 - g_2w - g_3}}$$

where the path of integration is the image under $\wp$ of a path from z_0 to z that avoids the zeros and poles of $\wp'(z)$, and where the sign of the square root must be chosen so that it actually equals $\wp'(z)$.

We recall that we encountered the relationship between elliptic functions and elliptic integrals already in connection with the conformal mapping of rectangles and certain triangles (Chap. 6, Sec. 2).

*EXERCISES

The Weierstrass functions satisfy numerous identities which are best dealt with in an exercise section. They can be proved either by comparing two elliptic functions with the same zeros and poles (when σ-functions are involved), or by comparing elliptic functions with the same singular parts (when only $\wp$- and ζ-functions are involved). The following sequence of formulas is so arranged that we need to resort to this method only once.

1.

$$\wp(z) - \wp(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2} \tag{16}$$

(Use (14) to show that the right-hand member is a periodic function of z. Find the multiplicative constant by comparing the Laurent developments.)

2.

$$\frac{\wp'(z)}{\wp(z)-\wp(u)} = \zeta(z-u) + \zeta(z+u) - 2\zeta(z). \tag{17}$$

(Follows from (16) by taking logarithmic derivatives.)

3.

$$\zeta(z+u) = \zeta(z) + \zeta(u) + \frac{1}{2}\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}. \tag{18}$$

(This is a symmetrized version of (17).)

4. The addition theorem for the $\wp$-function:

$$\wp(z+u) = -\wp(z) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(z)-\wp'(u)}{\wp(z)-\wp(u)}\right)^2. \tag{19}$$

(Differentiation of (18) leads to a formula which contains $\wp''(z)$. It can be eliminated by (15) which gives $\wp'' = 6\wp^2 - \frac{1}{2}g_2$. Symmetrization yields (19). Observe that this is an algebraic addition theorem, for $\wp'(z)$ and $\wp'(u)$ can be expressed algebraically through $\wp(z)$ and $\wp(u)$.)

5. Prove

$$\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z).$$

6. Prove $\wp'(z) = -\sigma(2z)/\sigma(z)^4$.

7. Prove that

$$\begin{vmatrix} \wp(z) & \wp'(z) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(u+z) & -\wp'(u+z) & 1 \end{vmatrix} = 0.$$

3.4. The Modular Function $\lambda(\tau)$. The differential equation (15) can also be written as

$$\wp'(z)^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3), \tag{20}$$

where e_1, e_2, e_3 are the roots of the polynomial $4w^3 - g_2w - g_3$.

To find the values of the e_k we determine the zeros of $\wp'(z)$. The symmetry and periodicity of $\wp(z)$ imply $\wp(\omega_1 - z) = \wp(z)$. Hence $\wp'(\omega_1 - z) = -\wp'(z)$, from which it follows that $\wp'(\omega_1/2) = 0$. Similarly $\wp'(\omega_2/2) = 0$, and also $\wp'((\omega_1+\omega_2)/2) = 0$. The numbers $\omega_1/2$, $\omega_2/2$ and $(\omega_1+\omega_2)/2$ are mutually incongruent modulo the periods. Therefore they are precisely the three zeros of $\wp'$, which is of order 3, and all the zeros are simple. When we compare with (20) it follows that we can set

$$e_1 = \wp(\omega_1/2), \qquad e_2 = \wp(\omega_2/2), \qquad e_3 = \wp((\omega_1+\omega_2)/2). \tag{21}$$